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Short Communication

A note on a variant of radial measure capable of dealing with negative inputs and outputs in DEA ☆

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ABSTRACT

The recent contribution by Cheng et al. (2013) presents a variant of the traditional radial input- and output-oriented efficiency measures whereby original values are replaced with absolute values. This comment spells out that this article contains some imprecisions and therefore presents some further results.

1. Introduction

The recently published article by Cheng, Zervopoulos, and Qian (2013) introduces a variant of the traditional input- or output-oriented radial efficiency measure in a DEA production model to handle negative values of inputs and outputs. However, this article contains some imprecisions in that the consequences for the interpretation of these efficiency measures are ignored. We draw upon the analysis in the earlier article by Kerstens and Van de Woestyne (2011) which treats the more general case of the directional distance function to refine some of these results.

In a traditional production context, inputs and outputs are assumed to be non-negative (see, e.g., Färe, Grosskopf, & Lovell (1994, pp. 44–45) for conditions on the input and output data matrices). When data can be naturally negative (e.g., growth rates, etc.), then modifications must be made to the basic DEA production specifications. The small literature accommodating negative data in DEA models has been competently surveyed in Pastor and Ruiz (2007).

2. Technology on extended data domain, directional distance function, and radial efficiency measures

When inputs and/or outputs can be negative, the notion of a technology must be generalized on this extended data domain as follows:

$$T' = \{(x, y) \in \mathbb{R}^{p+q}; x \text{ can produce } y\}, \tag{1}$$

with the assumptions listed in Kerstens and Van de Woestyne (2011). The directional distance function is defined as follows.

Definition 2.1. For a given technology T', the directional distance function $D_{T'}$ is the function $D_{T'} : T' \times ((-\mathbb{R}^p_+) \times \mathbb{R}^q_+) \rightarrow \mathbb{R} \cup \{+\infty\}$ with

$$D_{T'}(z; g) = \sup_\delta\{\delta \in \mathbb{R} : z + \delta g \in T'\}.$$

Further comments can be found in Kerstens and Van de Woestyne (2011), amongst others.

For an arbitrary vector $v = (v_1, \ldots, v_t) \in \mathbb{R}^t$, let $|v| = (|v_1|, \ldots, |v_t|) \in \mathbb{R}^t_+$. Denote by p_x and p_y the projection into input and output space respectively. Then, using specific direction vectors leads to the following special cases.

Definition 2.2.

(a) The generalized proportional distance function: $G_{T'}(z) = D_{T'}(z; (-p_x(|z|), p_y(|z|)))$.
(b) The generalized input-oriented directional distance function: $\overline{D}^i_{T'}(z) = D_{T'}(z; (-p_x(|z|), 0))$.
(c) The generalized output-oriented directional distance function: $\overline{D}^o_{T'}(z) = D_{T'}(z; (0, p_y(|z|)))$.

Obviously, for semi-positive inputs and outputs, Definition 2.2 coincides with their well-known counterparts. Note that choice (a) is exactly the one proposed by Kerstens and Van de Woestyne (2011).

The proportional interpretation of this generalized proportional distance function G and its extension to the generalized input- and

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output-oriented directional distance functions follows from the following proposition:

Proposition 2.1. *For a given technology T', $z \in T'$ and some reflection invariant norm function $\| \ldots \|$, the following holds:*

(a) $G_{T'}(z) = \hat{\delta}^* = \frac{\|z^*-z\|}{\|z\|}$, with $z^* = z + \hat{\delta}^*(-p_x(|z|), p_y(|z|))$.
(b) $0 \leqslant G_{T'}(z) \leqslant 1$ *if at least one of the input dimensions is strictly positive.*
(c) $\overline{D}^i_{T'}(z) = \hat{\delta}^* = \frac{\|z^*-z\|}{\|p_x(z)\|}$, *with* $z^* = z + \hat{\delta}^*(-p_x(|z|), 0)$.
(d) $0 \leqslant \overline{D}^i_{T'}(z) \leqslant 1$ *if at least one of the input dimensions is strictly positive.*
(e) $\overline{D}^o_{T'}(z) = \hat{\delta}^* = \frac{\|z^*-z\|}{\|p_y(z)\|}$, *with* $z^* = z + \hat{\delta}^*(0, p_y(|z|))$.
(f) $\overline{D}^o_{T'}(z) \geqslant 0$.

The proof of (a) and (b) is given in Kerstens and Van de Woestyne (2011, Proposition 3.1). The other cases can be proven similarly.

In Cheng et al. (2013), the input and output Variant of the Radial Measure (*VRM*) is introduced in their expressions (9) and (10). Trivially, their relation with Definition 2.2 is given by:

$$VRM^i(z) = 1 - \overline{D}^i_{T'}(z), \quad (2)$$

$$VRM^o(z) = \frac{1}{1 + \overline{D}^o_{T'}(z)}. \quad (3)$$

Observe that for semi-positive inputs and outputs, the input and output Variant of the Radial Measure (*VRM*) coincide with their traditional counterparts.

Cheng et al. (2013, p. 102) explicitly claim that these *VRM* efficiency measures have a proportional interpretation in that these involve "applying the ratio of proportionate input decrease to the observed input value (input-oriented) or the ratio of proportionate output increase to the observed output value (output-oriented)". However, as we immediately show, this claim needs qualification.

Combining (2) and (3) and Proposition 2.1 directly leads to the following result:

Proposition 2.2. *For a given technology T' and $z \in T'$, it follows that:*

(a) $0 \leqslant VRM^i(z) \leqslant 1$ *if at least one of the input dimensions is strictly positive.*
(b) $0 \leqslant VRM^o(z) \leqslant 1$.

To illustrate Proposition 2.2, we compute VRM^i and VRM^o using the data reported in Table 1 of Cheng et al. (2013): a sample of 13 observations, with 2 inputs and 3 outputs. Note that the first input is strictly positive while the second input is semi-negative. We consider two cases. The first case uses all available inputs and outputs, while the second case only uses the second (semi-negative) input. The resulting efficiency measures are reported in Table 1. First, in both cases, VRM^o is located between 0 and 1. Second, only in the first case when the first input is strictly positive, VRM^i is bounded between 0 and 1. However, when only the second semi-negative input is maintained in the second case, then VRM^i no longer obeys this restriction. For instance, Table 1 even denotes an efficiency value of −7.8893 for observation 5.

Table 1
VRM efficiency measures.

DMU	Case 1: 2 Inputs		Case 2: 1 Input	
	VRM^i	VRM^o	VRM^i	VRM^o
1	0.9417	0.9060	−3.0014	0.6680
2	0.6812	0.7695	−0.5746	0.7235
3	1.0000	1.0000	1.0000	1.0000
4	0.5580	0.6837	−4.5595	0.6837
5	0.8628	0.7706	−7.8893	0.5900
6	0.8584	0.8613	−4.9225	0.6492
7	1.0000	1.0000	0.6205	0.7912
8	1.0000	1.0000	1.0000	1.0000
9	0.8113	0.9123	0.0000	0.9123
10	0.4540	0.7303	−1.3524	0.7225
11	1.0000	1.0000	1.0000	1.0000
12	0.5101	0.6450	−1.1353	0.6192
13	1.0000	1.0000	1.0000	1.0000

Note finally that this extended data domain can yield problems for certain technology specifications (e.g., the assumption of constant returns to scale: see Silva Portela, Thanassoulis, & Simpson (2004) for details). For the DEA model of Banker, Charnes, and Cooper (1984), the solutions provided allow to handle negative data (as illustrated in Kerstens & Van de Woestyne (2011) and here above). While Cheng et al. (2013) also seem to focus on the same production technology in their final expressions (9) and (10), there is ambiguity in their expressions (5)–(6) and (7)–(8) where the so-called variable returns to scale constraint is put between brackets and the text maintains that it can be dropped to obtain constant returns to scale (see p. 102). To the best of our knowledge, there is no solution available in the literature to handle negative data under the constant returns to scale assumption.

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